2050C Special Tutorial 4

5.1 and 5.2

- Definition. f defined on some $E \subset \mathbb{R}$ and $x_0 \in E$. f is continuous at x_0 if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$, $\forall x \in E, |x - x_0| < \delta$. Here δ depends on ε and x_0.

- Sequential Criterion: f is continuous at x_0 if and only if, $\forall x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

Sequential Criterion is very useful in proving discontinuity. For instance, consider

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Then, taking $x_n = 1/(2n+\frac{1}{2})\pi$, $x_n \to 0$ but $f(x_n) = 1 \not\to f(0) = 0$, so $x_0 = 0$ is a discontinuity point.

- Basic continuous functions

ⓐ polynomials, conti on $(-\infty, \infty)$

ⓑ rational fcns, conti on $\{x : q(x) \neq 0\}$ when $f = P(x)/q(x)$.

ⓒ $\sqrt[n]{f(x)}$ where f is conti and ≥ 0.

ⓓ $|f(x)|$ where f is conti

ⓔ $\sin x$, $\cos x$, conti on $(-\infty, \infty)$.

<u>Proposition</u> the sine function is continuous everywhere.

Pf. Let $h = x - x_0$,

$$\sin x - \sin x_0 = \sin(x_0 + h) - \sin x_0$$

$$= \sin x_0 (\cos h - 1) + \sin h \cos x_0 .$$

$$\begin{aligned} |\sin x - \sin x_0| &\le |\cos h - 1||\sin x_0| + |\sin h||\cos x_0| \\ &\le |\cos h - 1| + |\sin h| . \\ &\le |\cos h - 1| + |h| \end{aligned}$$

after using $0 \le \frac{\sin h}{h} \le 1$, $h \ne 0$. Next, using

$$\sin \frac{h}{2} = \sqrt{\frac{\cos h - 1}{2}} ,$$

$$|\cos h - 1| \le 2 \sin^2 \frac{h}{2} \le 2\left(\frac{h}{2}\right)^2 \le \frac{1}{2} h^2 .$$

$$\therefore \quad |\sin x - \sin x_0| \le \frac{1}{2} h^2 + |h|$$

By squeeze thm,

$$0 \le \lim_{h \to 0} |\sin x - \sin x_0| \le \lim_{h \to 0} \left(\frac{1}{2} h^2 + |h|\right) = 0 .$$

- Using the fact that continuity is preserved under $+, -, \times, \div$, and
- Using Chain Rule (Continuity is preserved under composition)

we verify continuity of many functions.

e.g. $\dfrac{\sin(1+x^2-5x)}{\sqrt{1+x^2+\frac{1}{2}|x|}}$, $\forall x \in (-\infty, \infty)$, etc.

- Finally, two notable functions.

(a) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ f is disconti everyhere.

(b) $h(x) = \begin{cases} 1/q, & x = p/q,\ (p,q)=1, \\ 0, & x \notin \mathbb{Q} \cap (0,1) \end{cases}$ h is conti at $x \notin \mathbb{Q} \cap (0,1)$ but discont at $x \in \mathbb{Q} \cap (0,1)$.